

# Vacuum Fluctuations as Quantum Probes in FRWL space-times

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## Abstract

Vacuum fluctuations related a massless conformally coupled scalar field in Friedman-Robertson-Walker-Lemaitre (FRWL) space-times are investigated. Point-slitting regularization is used and a specific renormalization proposal is discussed. Applications to generic black holes and FRWL form of the de Sitter space-time are presented.

## 1 Introduction

How to probe the properties of the Quantum Vacuum in relativistic quantum field theory (QFT) ? It is well known that this is a difficult task, because one is dealing with relativistic systems having an infinite number of degrees of freedom. The possibility we shall discuss here is to study the vacuum expectation value  $\langle \phi^2(x) \rangle = \langle 0 | \phi^2(x) | 0 \rangle$  associated with quantum field  $\phi(x)$ . We will refer to it as “vacuum fluctuation”. It contains physical informations on the quantum vacuum, and it is simpler than the vacuum expectation value of the stress-energy tensor.

In QFT,  $\phi(x)$  is an operator valued distribution, thus  $\langle \phi^2(x) \rangle$  ill-defined quantity, and regularization and renormalization procedures are required.

One may use zeta-function regularization [1, 2], however, since in cosmology one is dealing with FRWL space-times and hyperbolic operators, point-splitting regularization is more appropriate.

Given the off-diagonal Wightman function

$$W(x, x') = \langle 0 | \phi(x) \phi(x') | 0 \rangle, \quad (1.1)$$

point-splitting regularization means  $x' = x + \varepsilon$ ,  $\varepsilon$  a small cut-off, and the removal of cut-off consists in taking the coincidence limit  $x' \rightarrow x$ .

## 2 Fluctuation as Quantum thermometer

Fluctuations may give informations on thermal nature of the quantum vacuum. In order to illustrate the issue, let us consider a free massive scalar field defined on the Minkowsky manifold  $M^4$ , make a Wick rotation to imaginary compactified time with period  $\beta$ . As a result, one is dealing with a massive free quantum field in thermal equilibrium at temperature  $T = \frac{1}{\beta}$ , and the relevant operator is defined on  $S_1 \times R^3$  and reads

$$L = -\partial_\tau^2 - \nabla^2 + M^2, \quad (2.1)$$

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$M$  being the mass and  $\tau$  imaginary time with period  $\beta$ . A direct calculation within zeta-function regularization [3, 4] gives for the renormalized fluctuation associated with this massive free field

$$\langle \phi(x)^2 \rangle_R = \frac{M}{2\pi\beta} \sum_{n=1}^{\infty} \frac{K_1(n\beta M)}{n} + \frac{M^2}{8\pi^2} \ln \left( \frac{M^2}{\mu^2} \right). \quad (2.2)$$

If  $M \neq 0$ , the thermal properties are not transparent and an arbitrary mass scale  $\mu^2$  is present: bad thermometer.

However, in the massless case, there is a drastic simplification and the result is

$$\langle \phi(x)^2 \rangle_R = \frac{1}{12\beta^2} = \frac{T^2}{12}. \quad (2.3)$$

No ambiguity is present, simple reading for the temperature  $T$ : good thermometer [5].

It is also instructive to present a point-splitting regularization derivation. Making use of the images method and KMS condition, it follows that thermal Wightman function is periodic of period  $\beta$  in the imaginary time  $\tau$ , and reads

$$W_\beta(x, x') = \frac{1}{4\pi^2} \sum_n \frac{1}{|\vec{x} - \vec{x}'|^2 + (\tau - \tau' + n\beta)^2}. \quad (2.4)$$

The term  $n = 0$  is the only singular term when  $x \rightarrow x'$ , and coincides with the Minkowski contribution. Renormalization prescription : subtract this term. Thus, for  $x \rightarrow x'$ , we get again

$$\langle \Phi(x)^2 \rangle = \frac{1}{2\pi^2\beta^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{12\beta^2} = \frac{T^2}{12}. \quad (2.5)$$

### 3 Spherically symmetric dynamical black holes

Fluctuations associated with massless scalar fields work well as good thermometers in finite temperature QFT in Minkowski space-time. What about the use of fluctuation as thermometers in presence of gravity?

Static black holes and FRWL space-times in cosmology are example of Spherically Symmetric Space-times (SSS), and both may have an unified descriptions [6]. With regard to this, first let us briefly review the Kodama-Hayward formalism. A generic dynamical SSS may be defined by the following metric

$$ds^2 = \gamma_{ij}(x^i) dx^i dx^j + R^2(x^i) d\Omega^2, \quad i, j = 0, 1, \quad (3.1)$$

where the two dimensional *normal metric* is

$$d\gamma^2 = \gamma_{ij}(x^i) dx^i dx^j, \quad (3.2)$$

with  $x^i$  associated coordinates. Furthermore, the quantity  $R(x^i)$  is the areal radius, a scalar field in the normal 2-dimensional metric. Dynamical or apparent trapping horizons are determined by

$$\gamma^{ij} \partial_i R_H \partial_j R_H = 0. \quad (3.3)$$

Another important invariant quantity is the Hayward surface gravity  $\kappa_H = \frac{1}{2}(\Delta_\gamma R)_H$  (see for example [7, 8]), which is a generalization of the Killing surface gravity. There is also conserved

Kodama vector,  $K^i = \frac{\varepsilon^{ij} \partial R_j}{\sqrt{-\gamma}}$ , generalization of the Killing vector. Relevant for us are the so called Kodama observers, defined by condition  $R = R_0$ .

First Example: 4-dimensional static Schwarzschild BH with  $x = (t, r)$  as coordinates in normal space.

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega^2 = d\gamma^2 + r^2 d\Omega^2 \quad (3.4)$$

with

$$V(r) = 1 - \frac{2M}{r}, \quad G = 1 \quad (3.5)$$

Areal radius  $R = r$ , the event horizon  $V(r_H) = 0$ , i.e.  $r_H = 2M$ , the surface gravity  $\kappa_H = \frac{V'_H}{2} = \frac{1}{4M}$ , Kodama vector is the Killing vector  $K = (1, 0, 0, 0)$ . Kodama observer:  $r = r_0$ : constant areal radius.

A dynamical SSS example: flat FRWL space-time, relevant in cosmology

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2), \quad (3.6)$$

Areal radius is  $R = a(t)r$ , physical radial distance,  $H(t) = \frac{\dot{a}}{a}$  is the Hubble parameter. There is a dynamical trapping horizon (Hubble radius)

$$R_H = \frac{1}{H(t)}. \quad (3.7)$$

The Kodama vector is

$$K = (1, -H(t)R, 0, 0) \quad (3.8)$$

Kodama observer:  $R = R_0$  constant areal radius, namely  $r = \frac{R_0}{a(t)}$ .

## 4 Fluctuations in FRWL space-times

The main idea: use the fluctuation related to a suitable quantum probe, which has to be a good thermometer. The probe: a conformally coupled massless scalar field. Its Lagrangian reads

$$L = \sqrt{-g} \left( -\frac{1}{2} \partial^i \phi \partial_i \phi - \frac{1}{6} \mathcal{R} \phi^2 \right), \quad (4.1)$$

$\mathcal{R}$  Ricci scalar curvature.

It is convenient to re-write the flat FRWL space-time making us of the conformal time  $\eta$ ,  $d\eta = \frac{dt}{a}$ ,

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2). \quad (4.2)$$

Recall that the Wightmann function is  $W(x, x') = \langle 0 | \phi(x) \phi(x') | 0 \rangle$ . Now the field  $\phi(x)$  is “free” and admits the expansion

$$\phi(x) = \sum_{\vec{k}} f_{\vec{k}}(x) a_{\vec{k}} + h.c.. \quad (4.3)$$

where  $f_{\vec{k}}(x)$  are the associated modes. The related Conformal Vacuum is defined by

$$a_{\vec{k}}|0\rangle = 0. \quad (4.4)$$

As a result

$$W(x, x') = \sum_{\vec{k}} f_{\vec{k}}(x) f_{\vec{k}}^*(x'), \quad (4.5)$$

where the modes functions  $f_{\vec{k}}(x)$  satisfy conformally invariant equations

$$\left(\Delta - \frac{\mathcal{R}}{6}\right) f_{\vec{k}}(x) = 0. \quad (4.6)$$

In this case, solutions of this equation are simple and characterize the Conformal Vacuum:

$$f_{\vec{k}}(x) = \frac{e^{-i\eta k}}{2\sqrt{k}a(\eta)} e^{-i\vec{k}\cdot\vec{x}} \quad k = |\vec{k}| \quad (4.7)$$

The related Wightmann function  $W(x, x')$  can be evaluated, and this covariant bi-scalar distribution reads

$$W(x, x') = \frac{1}{4\pi^2 a(\eta) a(\eta')} \frac{1}{|\vec{x} - \vec{x}'|^2 - |\eta - \eta' - i\epsilon|^2}. \quad (4.8)$$

It is convenient to make use of the proper time  $\tau$  as evolution parameter. As a result, the proper-time-parametrized Wightman function is

$$W(x(\tau), x'(\tau')) = \frac{1}{4\pi^2} \frac{1}{\sigma^2(\tau, \tau')}, \quad (4.9)$$

where the invariant distance is defined by

$$\sigma^2(\tau, \tau') = a(\tau) a(\tau') \left(x(\tau) - x(\tau')\right)^2. \quad (4.10)$$

## 5 Point-splitting regularization

Due to the isotropy of flat FRWL space-time, one may restrict only to radial time-like trajectories, namely  $x(\tau) = (\eta(\tau), r(\tau))$ .

Putting  $\tau' = \tau + \varepsilon$ , with  $\varepsilon$  small and  $\dot{t} = \frac{dt}{d\tau}$ , and introducing the four-acceleration along the trajectory from  $x$  to  $x'$ ,

$$A^2 = \left[ \frac{\ddot{t}}{\sqrt{\dot{t}^2 - 1}} + H \sqrt{\dot{t}^2 - 1} \right]^2, \quad (5.1)$$

one obtains

$$\sigma^2(\tau, \varepsilon) = -\varepsilon^2 - \frac{1}{12} \left[ A^2 + H^2 + 2\dot{t}^2 \partial_t H \right] \varepsilon^4 + O(\varepsilon^6). \quad (5.2)$$

As a result, for small  $\varepsilon$  the Wightman function is

$$W(\tau, \varepsilon) = -\frac{1}{4\pi^2} \frac{1}{\varepsilon^2} + \frac{1}{48\pi^2} \left[ A^2 + H^2 + 2\dot{t}^2 \partial_t H \right] + O(\varepsilon^2). \quad (5.3)$$

## 6 Renormalization

In Minkowski space-time, one has  $H(t) = 0$ , and for inertial trajectories, one has vanishing acceleration, namely

$$W_M(\tau, \varepsilon) = -\frac{1}{4\pi^2} \frac{1}{\varepsilon^2}. \quad (6.1)$$

Choice of the renormalization prescription: subtract this contribution. As a consequence, the renormalized vacuum fluctuation reads [6]

$$\langle \phi^2(x) \rangle_R = \frac{1}{48\pi^2} [A^2 + H^2 + 2\dot{t}^2 \partial_t H], \quad (6.2)$$

Some remarks are in order. The first term is depending on the trajectory through  $A$  and hence  $\dot{t}$ . The second one is depending on the dynamical space-time through  $H$ , and the third term is an “entangled” contribution, vanishing for stationary space-times.

Important byproduct. In Minkowski case,  $H = 0$ . Thus one has only two cases:

If  $A$  not vanishing: one has the Unruh effect, and the related Unruh temperature is

$$T_U = \frac{A}{2\pi}. \quad (6.3)$$

On the other hand, if  $A = 0$ , namely inertial observer, one has  $\langle \phi^2(x) \rangle_R = 0$ , and the Minkowski renormalized result is obtained.

## 7 de Sitter FRWL space-time

A very important example of FRWL space-time is de Sitter one, its flat FRWL metric being

$$ds^2 = -dt^2 + e^{2H_0 t} d\vec{x}^2, \quad (7.1)$$

$a(t) = e^{H_0 t}$ ,  $H(t) = H_0$  constant.

For Kodama observers with  $R = R_0$ , the conformal time can be evaluated in terms of proper time  $\tau$

$$\eta(\tau) = \frac{1}{H_0} e^{-\frac{H_0}{\sqrt{1-R_0^2 H_0^2}} \tau} \quad (7.2)$$

and the dS invariant distance is

$$\sigma^2(s) = -\frac{(1-R_0^2 H_0^2)}{4H_0^2} \sinh^2 \left( \frac{H_0}{2\sqrt{1-R_0^2 H_0^2}} s \right) \quad (7.3)$$

This is an example of stationary space-time: it depends only on the difference  $s = \tau - \tau'$ .

General formula (6.2) leads to the fluctuation for dS

$$\langle \phi^2(x) \rangle_R = \frac{1}{48\pi^2} \left( \frac{H_0^4 R_0^2}{(1-R_0^2 H_0^2)} + H_0^2 \right), \quad (7.4)$$

the first term in the bracket being the acceleration of the Kodama observer at fixed  $R_0$ . One also has

$$\langle \phi^2(x) \rangle_R = \frac{1}{12} \frac{T_{GH}^2}{(1-R_0^2 H_0^2)}. \quad (7.5)$$

$T_{GH} = \frac{H_0}{2\pi}$  is the well known Gibbons-Hawking temperature,  $\frac{1}{(1-R_0^2 H_0^2)}$  being a kinematical red-shift factor. One is dealing with a good thermometer.

## 8 de Sitter space as black holes

The de Sitter space-time admits also static patch

$$ds^2 = -(1 - H_0^2 r^2) dt_s^2 + \frac{dr^2}{1 - H_0^2 r^2} + r^2 d\Omega^2, \quad (8.1)$$

$t_s$  is the time coordinate, and the areal radius is  $R = r$ . The related horizon:  $r_H = \frac{1}{H_0}$ , and surface gravity  $\kappa_H = H_0$ .

In the cosmological dS patch, the fluctuation leads, modulo a red-shift factor, to the Gibbons-Hawking temperature  $T_H = \frac{H_0}{2\pi}$ . What about the temperature issue in this static patch? One can answer to this question in general.

## 9 Static black hole as effective FRWL space-time

Black holes are not black, and it is well known that they emit a quantum Hawking radiation in thermodynamical equilibrium at temperature  $T_H$ . In order to (partially) investigate this fundamental issue, first let us show that for Kodama observers, one is dealing with a special FRWL space-time. Start with a generic static BH solution

$$ds^2 = -V(r) dt_s^2 + \frac{dr^2}{V(r)} + r^2 d\Omega^2 \quad (9.1)$$

with event horizon at  $V(r_H) = 0$ ,  $V'_H \neq 0$ . In order to avoid the metric singularity at  $r = r_H$ , one can make use of Kruskal coordinates. With tortoise radial coordinate  $dr^* = \frac{dr}{V(r)}$ , the BH Kruskal metric is

$$ds^2 = e^{-2\kappa_H r^*} V(r^*) [-dT^2 + dX^2] + r^2(T, R) d\Omega^2, \quad (9.2)$$

where now  $r^* = r^*(T, X)$ ,  $\kappa_H = \frac{V'_H}{2}$  being the Killing surface gravity. The relevant part of BH Kruskal metric is its normal part, namely

$$d\gamma^2 = e^{-2\kappa_H r^*} V(r^*) [-dT^2 + dX^2]. \quad (9.3)$$

First key point : Kruskal normal space-time is conformally related to two-dimensional Minkowski space-time.

Second point: Kruskal space-time is effective flat FRWL space-time for Kodama observers  $r = r_0$

$$d\gamma^2 = V_0 e^{-2\kappa_H r_0^*} (-dT^2 + dX^2), \quad (9.4)$$

and we also have

$$d\gamma^2 = -dt^2 + a_*^2 dX^2 = -dt_s^2 V_0. \quad (9.5)$$

Thus  $t = \sqrt{V_0} e^{-\kappa_H r_0^*} T$  is a “cosmological” time, and  $a(r_0^*) = \sqrt{V_0} e^{-\kappa_H r_0^*}$  is the related constant “expansion” factor, and  $d\tau^2 = dt_s^2 V_0$ .

Since  $a^2(r_0)$  constant, i.e.  $H = 0$ , only acceleration term is present, and the general formula (6.2) gives

$$\langle \phi^2(x) \rangle_R = \frac{1}{48\pi^2} \left[ \frac{(\ddot{t})^2}{\dot{t}^2 - 1} \right]. \quad (9.6)$$

Using Kruskal coordinates definition and  $t_s = \frac{\tau}{\sqrt{V_0}}$ , one gets

$$\dot{t} = \cosh\left(\kappa_H \frac{\tau}{\sqrt{V_0}}\right), \quad \ddot{t} = \frac{\kappa_H}{\sqrt{V_0}} \sinh\left(\kappa_H \frac{\tau}{\sqrt{V_0}}\right). \quad (9.7)$$

As a result

$$\langle \phi^2(x) \rangle_R = \frac{1}{48\pi^2} \frac{\kappa_H^2}{V_0} = \frac{1}{12} \frac{T_H^2}{V_0}. \quad (9.8)$$

Again a good thermometer: local temperature is

$$T_0 = \frac{T_H}{\sqrt{V_0}}, \quad (9.9)$$

with the Hawking temperature  $T_H = \frac{\kappa_H}{2\pi}$ , while  $V_0 = -g_{00}$  being the Tolman red-shift factor. In the de Sitter space-time,  $V(r) = 1 - H_0^2 r^2$  and  $T_H = \frac{|\kappa_H|}{2\pi} = \frac{H_0}{2\pi}$ , in full agreement with previous result.

## 10 Conclusion

Fluctuation related to massless conformally coupled scalar field has been used to probe the Quantum Vacuum in presence of gravity. Making use of this quantum probe, after a point-splitting regularization in the proper-time, the following renormalization prescription has been used: remove the Minkowski contribution related to inertial observers. In principle, other renormalization prescriptions are possible. However, one can show that our renormalization prescription is full agreement with the results obtained making use of Unruh- de Witt detector approach [9].

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